TIME-FREQUENCY ANALYSIS OF SJÖSTRAND'S CLASS

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ABSTRACT. We investigate the properties an exotic symbol class of pseudodifferential operators, Sjöstrand's class, with methods of time-frequency analysis (phase space analysis). Compared to the classical treatment, the time-frequency approach leads to striklingly simple proofs of Sjöstrand's fundamental results and to far-reaching generalizations.

1. INTRODUCTION

In 1994/95 Sjöstrand introduced a symbol class for pseudodifferential operators that contains the Hörmander class $S^0_{0,0}$ and also includes non-smooth symbols. He proved three fundamental results about the L^2- boundedness, the algebra property, and the Wiener property. This work had considerable impact on subsequent work in both hard analysis [9, 10, 28, 42–44] and time-frequency analysis [11, 23, 24].

Sjöstrand's definition goes as follows: Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be a C^∞-function with compact support satisfying the property $\sum_{k \in \mathbb{Z}^{2d}} g(t-k) = 1, \forall t \in \mathbb{R}^{2d}$. Then a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ belongs to $M^{\infty,1}$, the Sjöstrand class, if

$$\int_{\mathbb{R}^{2d}} \sup_{k \in \mathbb{Z}^{2d}} |(\sigma \cdot g(.\,-k))\hat{}(\zeta)| \, d\zeta < \infty \,.$$

The Weyl transfrom of a symbol $\sigma(z, \zeta)$ is defined as

$$(1) \qquad \sigma^w f(x) = \int_{\mathbb{R}^d} \sigma\big(\frac{x+y}{2}, \xi\big) e^{2\pi i (x-y) \cdot \xi} f(y) \, dy d\xi \,.$$

Sjöstrand proved the following fundamental results about the Weyl transform of a symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ [38, 39].

(a) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then σ^w is a bounded operator on $L^2(\mathbb{R}^d)$.

(b) If $\sigma_1, \sigma_2 \in M^{\infty,1}(\mathbb{R}^{2d})$ and $\tau^w = \sigma_1^w \sigma_2^w$, then $\tau \in M^{\infty,1}(\mathbb{R}^{2d})$; thus $M^{\infty,1}$ is a (Banach) algebra of pseudodifferential operators.

(c) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and σ^w is invertible on $L^2(\mathbb{R}^d)$, then $(\sigma^w)^{-1} = \tau^w$ for some $\tau \in M^{\infty,1}(\mathbb{R}^{2d})$. This is the Wiener property of $M^{\infty,1}$. For the classical symbol classes results of this type go back to Beals [3].

The original proofs of Sjöstrand were carried out in the realm of classical "hard" analysis. This line of investigation was deepened and extended in subsequent work by Boulkhemair, Herault, and Toft [9, 10, 28, 42–44].

Later it was discovered that Sjöstrand's class $M^{\infty,1}$ is a special case of a so-called modulation space. The family of modulation spaces had be studied in time-frequency analysis since the 1980s and later was also used in the theory of pseudodifferential operators. The action of pseudodifferential operators with classical symbols on modulation spaces was investigated by Tachizawa [41] in 1994; general modulation spaces as symbol classes for pseudodifferential operators were introduced in [23] independently of Sjöstrand's work. This line of investigation and the emphasis on time-frequency techniques was continued in [11, 12, 23, 24, 31, 32].

To make the connection to time-frequency analysis, we introduce the operators of translation and modulation,

$$(2) \qquad T_x f(t) = f(t - x) \qquad \text{and} \qquad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t), \qquad t, x, \omega \in \mathbb{R}^d,$$

and note that

$$(3) \qquad (\sigma \cdot g(\cdot - z))\widehat{}(\zeta) = \int_{\mathbb{R}^{2d}} \sigma(t) \bar{g}(t - z) \, e^{-2\pi i \zeta \cdot t} \, dt = \langle \sigma, M_\zeta T_z g \rangle \, .$$

This is the so-called short-time Fourier transform. It is not only an important and widely used time-frequency representation in signal analysis, but an important object in the mathematical theory of time-frequency analysis. A physicist would use a different terminology for the same object and speak of position z, momentum ζ, and phase space $\mathbb{R}^{2d}$ instead of time and frequency.

In view of (3) a distribution belongs to Sjöstrand's class, if its short-time Fourier transform satisfies the condition $\int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} |\langle \sigma, M_\zeta T_x g \rangle| \, d\zeta < \infty$. More generally, the modulation spaces are defined by imposing a weighted L^p-norm on the short-time Fourier transform. This class of function spaces was introduced by H. G. Feichtinger in 1983 [15] and [14, 16] and has been studied extensively. The modulation spaces have turned out to be the appropriate function and distribution spaces for many problems in time-frequency analysis.

The objective of this paper is to give the "natural" proofs of Sjöstrand's results. The definition of Sjöstrand's by means of the short-time Fourier transform (3) suggests that the mathematics of translation and modulation operators, in other words, time-frequency analysis, should enter in the proofs. Although "natural" is a debatable notion in mathematics, we argue that methods of time-frequency analysis should simplify the original proofs and shed new light on Sjöstrand's results. Currently, several different proofs exist for the boundedness and the algebra property, both in the context of "hard analysis" and of time-frequency analysis. However, for the Wiener property only Sjöstrand's original "hard analysis" proof was known, and it was an open problem to find an alternative proof.

In the following, we will not only give conceptually new and technically simple proofs of Sjöstrand's fundamental results, but we will also obtain new insights.

Firstly, time-frequency methods provide detailed information on which class of function spaces Weyl transforms with symbols in $M^{\infty,1}$ act boundedly.

Secondly, the time-frequency methods suggest the appropriate and maximal generalization of Sjöstrand's results (to weighted modulation spaces). Although we restrict our attention to Weyl transforms and modulation spaces on $\mathbb{R}^d$, all concepts

can be defined on arbitrary locally compact abelian groups. One may conjecture that Sjöstrand's results hold for (pseudodifferential) operators on L^2 of locally compact abelian groups as well. In that case time-frequency methods hold more promise than real analysis methods.

Thirdly, we show that Weyl transforms with symbols in Sjöstrand's class are almost diagonalized by Gabor frames. This may not be surprising, because it is well-known that pseudodifferential operators with classical symbols are almost diagonalized by wavelet bases and local Fourier bases [33, 35]. What is remarkable is that the almost diagonalization property with respect to Gabor frames is a characterization of Sjöstrand's class.

Finally, the new proof of the Wiener property highlights the interaction with recent Banach algebra techniques, in particular the functional calculus in certain matrix algebras.

The remainder of the paper is divided into three parts. In Section 2 we introduce the basic definitions and results from time-frequency analysis. This area has now reached a level of sophistication that makes it possible to approach a subject that is usually the domain of "hard analysis". In Section 3 we prove the almost diagonalization property of pseudodifferential operators with symbols in Sjöstrand's class. In Section 4 we prove Sjöstrand's results and their generalization. With the background in time-frequency analysis this part becomes very short. We conclude with some remarks and problems.

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2. Tools from Time-Frequency Analysis

We prepare the tools from time-frequency analysis. Most of these are standard and discussed at length in the text books [18, 21], but the orginal ideas go back much further.

2.1. **Time-Frequency Representations.** We combine time $x \in \mathbb{R}^d$ and frequency $\xi \in \mathbb{R}^d$ into a single point $z = (x, \xi)$ in the "time-frequency" plane $\mathbb{R}^{2d}$. Likewise we combine the operators of translation and modulation to a *time-frequency shift* and write

$$\pi(z)f(t) = M_\xi T_x f(t) = e^{2\pi i \xi \cdot t} f(t - x)$$

The *short-time Fourier transform (STFT)* of function/distribution f on $\mathbb{R}^d$ with respect to window g is defined by

$$
\begin{aligned}
V_g f(x, \omega) &= \int_{\mathbb{R}^d} f(t) \bar{g}(t - x) e^{-2\pi i t \cdot x} \, dt \\
&= \langle f, M_\xi T_x g \rangle = \langle f, \pi(z)g \rangle.
\end{aligned}
$$

The short-time Fourier transform of a symbol $\sigma(x,\xi), (x,\xi) \in \mathbb{R}^{2d}$, is a function on $\mathbb{R}^{4d}$ and will be denoted by $\mathcal{V}_\Phi \sigma(z,\zeta)$ for $z,\zeta \in \mathbb{R}^{2d}$ in order to distinguish it from the STFT of a function on $\mathbb{R}^d$.

Usually we fix g in a space of test functions, e.g., $g \in \mathcal{S}(\mathbb{R}^d)$, and interpret $f \to V_g f$ as a linear mapping and $V_g f(x,\xi)$ as the time-frequency content of f near the point (x,ξ) in the time-frequency plane.

Similarly, the (cross-) *Wigner distribution* of $f,g \in L^2(\mathbb{R}^d)$ is defined as

$$W(f,g)(x,\xi) = \int f(x + \frac{t}{2})\overline{g(x - \frac{t}{2})}e^{-2\pi i \xi t}\, dt.$$

Writing $\check{g}(t) = g(-t)$ for the inversion, we find that the Wigner distribution is just a short-time Fourier transform in disguise:

$$W(f,g)(x,\xi) = 2^d\, e^{4\pi i x \cdot \xi}\, V_{\check{g}} f(2x, 2\xi)\,.$$

We will need a well-known intertwining property of Wigner distribution, which expresses the Wigner distribution of a time-frequency shift as a time-frequency shift, see [18, p. 57] and [21, Prop. 4.3.2].

Lemma 2.1. *Let $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{R}^{2d}$ and $f,g \in L^2(\mathbb{R}^d)$. Then*

$$W(\pi(w)f, \pi(z)g)(x,\xi)$$
$$= e^{\pi i(z_1+w_1)\cdot(z_2-w_2)}\, e^{2\pi i x \cdot(w_2-z_2)}\, e^{2\pi i \xi \cdot(-w_1+z_1)} \cdot W(f,g)(x - \frac{w_1+z_1}{2}, \xi - \frac{w_2+z_2}{2})\,.$$

In short, with the notation $j(z) = j(z_1, z_2) = (z_2, -z_1)$ we have

$$(4) \qquad\qquad W(\pi(w)f, \pi(z)g) = c M_{j(w-z)} T_{\frac{w+z}{2}} W(f,g)\,,$$

and the phase factor $c = e^{\pi i(z_1+w_1)\cdot(z_2-w_2)}$ is of modulus 1.

2.2. **Weyl Transforms.** Using the Wigner distribution, we can recast the definition of the Weyl transform as follows:

$$(5) \qquad\qquad \langle \sigma^w f, g \rangle = \langle \sigma, W(g,f) \rangle \qquad f,g \in \mathcal{S}(\mathbb{R}^d)$$

In the context of time-frequency analysis this is the appropriate definition of the Weyl transform, and we will never use the explicit formula (1). Whereas the integral in (1) is defined only for a restricted class of symbols (σ should be locally integrable at least), the time-frequency definition of σ^w makes sense for arbitrary $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. In addition, if $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is continuous, then the Schwartz kernel theorem implies that there exists a $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $\langle Tf, g \rangle = \langle \sigma^w f, g \rangle$ for all $f,g \in \mathcal{S}(\mathbb{R}^d)$. Thus, in a distributional sense, every reasonable operator possesses a Weyl symbol.

The composition of Weyl transforms defines bilinear form on symbols (*twisted product*)

$$\sigma^w \tau^w = (\sigma \,\sharp\, \tau)^w$$

Again, there is a (complicated) explicit formula for the twisted product [18, 29], but it is unnecessary for our purpose.

2.3. **Weight Functions.** We use two classes of weight functions. By v we always denote a non-negative function on $\mathbb{R}^{2d}$ with the following properties:
(i) v is continuous, $v(0) = 1$, and v is even in each coordinate $v(\pm z_1, \pm z_2, \ldots, \pm z_{2d}) = v(z_1, \ldots, z_{2d})$,
(ii) v is submultiplicative, i.e., $v(w + z) \leq v(w)v(z), w, z \in \mathbb{R}^{2d}$,
(iii) v satisfies the GRS-condition (Gelfand-Raikov-Shilov [20])

$$(6) \qquad \lim_{n \to \infty} v(nz)^{1/n} = 1, \qquad \forall z \in \mathbb{R}^{2d}.$$

We call a weight satisfying properties (i) — (iii) *admissible*. Every weight of the form $v(z) = e^{a|z|^b}(1 + |z|)^s \log^r(e + |z|)$ for parameters $a, r, s \geq 0$, $0 \leq b < 1$ is admissible, whereas the exponential weight $v(z) = e^{a|z|}, a > 0$, is not, because it violates (6).

Associated to an admissible weight v, we define the class of so-called v-*moderate* weights by

$$(7) \qquad \mathcal{M}_v = \{m \geq 0 : \sup_{w \in \mathbb{R}^{2d}} \frac{m(w + z)}{m(w)} \leq Cv(z), \forall z \in \mathbb{R}^{2d}\}.$$

Compare also [29, Ch. 18.5]. This definition implies that the weighted mixed-norm ℓ^p-space $\ell_m^{p,q}$ is invariant under translation whenever $m \in \mathcal{M}_v$. Precisely, set

$$\|c\|_{\ell_m^{p,q}} = \left(\sum_{l \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |c_{kl}|^p \, m(\alpha k, \beta l)^p \right)^{q/p} \right)^{1/q},$$

and $(T_{(r,s)}c)_{(k,l)} = c_{(k-r,l-s)}, k, l, r, s \in \mathbb{Z}^d$, then $\|T_{(r,s)}c\|_{\ell_m^{p,q}} \leq Cv(\alpha r, \beta s)\|c\|_{\ell_m^{p,q}}$. Consequently, Young's theorem for convolution implies that $\ell_v^1 * \ell_m^{p,q} \subseteq \ell_m^{p,q}$.

2.4. **Modulations Spaces and Symbol Classes.** Let $\varphi(t) = e^{-\pi t^2}$ be the Gaussian on $\mathbb{R}^d$, then we define a norm on f by imposing a norm on the short-time Fourier transform of f as follows:

$$\|f\|_{M_m^{p,q}} = \|V_\varphi f\|_{L_m^{p,q}}$$
$$= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p \, m(x, \xi)^p \, dx \right)^{q/p} d\xi \right)^{1/q}$$

If $1 \leq p, q < \infty$ and $m \in \mathcal{M}_v$, we define $M_m^{p,q}(\mathbb{R}^d)$ as the completion of the subspace $\mathcal{H}_0 = \text{span}\{\pi(z)\varphi : z \in \mathbb{R}^{2d}\}$ with respect to this norm, if $p = \infty$ or $q = \infty$, we use a weak-* completion. For $p = q$ we write M_m^p for $M_m^{p,p}$, for $m \equiv 1$, we write $M^{p,q}$ instead of $M_m^{p,q}$. For the theory of modulation spaces and some applications we refer the reader to [16] and [21, Ch. 11-13]
REMARKS: 1. The cautionary definition is necessary only for weights of super-polynomial growth. If $m(z) = \mathcal{O}(|z|^N)$ for some $N > 0$, then $M_m^{p,q}$ is in fact the subspace of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which $\|f\|_{M_m^{p,q}}$ is finite. If $m \geq 1$ and $1 \leq p, q \leq 2$, then $M_m^{p,q}$ is a subspace of $L^2(\mathbb{R}^d)$. However, if $v(z) = e^{a|z|^b}, b < 1$, then $M_v^1 \subseteq \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d) \subseteq M_{1/v}^\infty$, and we would have to use ultradistributions in the sense of Björk [6] to define $M_m^{p,q}$ as a subspace of "something".

2. *Equivalent norms:* Assume that $m \in \mathcal{M}_v$ and that $g \in M_v^1$, then

$$(8) \qquad \qquad \|V_g f\|_{L_m^{p,q}} \asymp \|f\|_{M_m^{p,q}} \,.$$

Therefore we can use arbitrary windows in M_v^1 in place of the Gaussian to measure the norm of $M_m^{p,q}$ [21, Ch. 11]. In the following we will use this norm equivalence frequently without mentioning.

3. The class of modulation spaces contains a number of classical function spaces [21, Prop.11.3.1], in particular $M^2 = L^2$; if $m(x, \xi) = (1 + |\xi|^2)^{s/2}, s \in \mathbb{R}$, then $M_m^2 = H^2$, the Bessel potential space; likewise, the Shubin class Q_s can be identified as a modulation space [7,37]; and even $\mathcal{S}$ can be represented as an intersection of modulation spaces.

4. If $m \in \mathcal{M}_v$, the following embeddings hold for $1 \leq p, q \leq \infty$:

$$M_v^1 \hookrightarrow M_m^{p,q} \hookrightarrow M_{1/v}^\infty \,,$$

and M_v^1 is dense in $M_m^{p,q}$ for $p, q < \infty$, and weak-* dense otherwise.

5. The original Sjöstrand class is $M^{\infty,1}(\mathbb{R}^{2d})$ [38, 39]. We will use the weighted class $M_v^{\infty,1}$ as a symbol class for pseudodifferential operators in our investigation. For explicitness, we recall the norm of $\sigma \in M_v^{\infty,1}$:

$$(9) \qquad \qquad \|\sigma\|_{M_v^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(z, \zeta)| \, v(\zeta) \, d\zeta \,.$$

In the last few years modulation spaces have been used implicitly and explicity as symbol classes by many authors, see [9–12, 23–25, 27, 28, 31, 32, 34, 35, 41–44] for a sample of work.

2.5. **Gabor Frames.** Fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$. Usually we take $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ or $\Lambda = \alpha \mathbb{Z}^{2d}$ for some $\alpha, \beta > 0$. Let $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ be the orbit of g under $\pi(\Lambda)$. Associated to $\mathcal{G}(g, \Lambda)$ we define two operators; first the coefficient operator C_g which maps functions to sequences on Λ and is defined by

$$(10) \qquad \qquad C_g f(\lambda) = \langle f, \pi(\lambda)g \rangle \,, \qquad \lambda \in \Lambda \,,$$

and then the Gabor frame operator $S = S_{g,\Lambda}$

$$(11) \qquad \qquad Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \, \pi(\lambda)g = C_g^* C_g f \,.$$

Definition 1. The set $\mathcal{G}(g, \Lambda)$ is called a Gabor frame (Weyl-Heisenberg frame) for $L^2(\mathbb{R}^d)$, if $S_{g,\Lambda}$ is bounded and invertible on $L^2(\mathbb{R}^d)$. Equivalently, C_g is bounded from $L^2(\mathbb{R}^d)$ into $\ell^2(\Lambda)$ with *closed range*, i.e. $\|f\|_2 \asymp \|C_g f\|_2$.

If $\mathcal{G}(g, \Lambda)$ is a frame, then the function $\gamma = S^{-1}g \in L^2(\mathbb{R}^d)$ is well defined and is called the (canonical) dual window. Likewise the "dual tight frame window" $\tilde{\gamma} = S^{-1/2}g$ is in $L^2(\mathbb{R}^d)$. Using different factorizations of the identity and the commutativity $S_{g,\Lambda}\pi(\lambda) = \pi(\lambda)S_{g,\Lambda}$ for all $\lambda \in \Lambda$, we obtain the following series

expansions (Gabor expansions) for $f \in L^2(\mathbb{R}^d)$:

$$
\begin{equation}
f \;=\; S^{-1}S = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \, \pi(\lambda)\gamma \tag{12}
\end{equation}
$$

$$
\begin{equation}
= \; SS^{-1}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \, \pi(\lambda)g \,. \tag{13}
\end{equation}
$$

$$
\begin{equation}
= \; S^{-1/2}SS^{-1/2}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)S^{-1/2}g \rangle \, \pi(\lambda)S^{-1/2}g \,. \tag{14}
\end{equation}
$$

The so-called "tight Gabor frame expansion" (14) is particularly useful and convenient, because it uses only one window $S^{-1/2}g$ and behaves like an orthonormal expansion (with the exception that the coefficients are not unique).

The existence and construction of Gabor frames for separable lattices $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ is well understood (see [13, 21, 30, 45]) and we may take the existence of Gabor frames with suitable g for granted.

The expansions (12) – (14) converge unconditionally in $L^2(\mathbb{R}^d)$, but for "nice" windows the convergence can be extended to other function spaces.

The following theorem summarizes the main properties of Gabor expansions and the characterization of time-frequency behavior by means of Gabor frames [17, 26]

Theorem 2.2. *Let v be an admissible weight function (in particular v satisfies the GRS-condition (6)). Assume that $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ and that $g \in M_v^1$. Then*

(i) The dual window $\gamma = S^{-1}g$ and $S^{-1/2}g$ are also in M_v^1.

(ii) If $f \in M_m^{p,q}$, then the Gabor expansions (12) – (14) converge unconditionally in $M_m^{p,q}$ for $1 \le p, q < \infty$ and all $m \in \mathcal{M}_v$, and weak-* unconditionally if $p = \infty$ or $q = \infty$.

(iii) The following norms are equivalent on $M_m^{p,q}$:

$$
\begin{equation}
\|f\|_{M_m^{p,q}} \asymp \|C_g f\|_{\ell_m^{p,q}} \asymp \|C_\gamma f\|_{\ell_m^{p,q}} \,. \tag{15}
\end{equation}
$$

REMARK: When $g \in M^1 \supseteq M_v^1$, then $\mathcal{G}(g, \Lambda)$ is necessarily overcomplete by the Balian–Low theorem [4]. Although the coefficients $\langle f, \pi(\lambda)g \rangle$ and $\langle f, \pi(\lambda)\gamma \rangle$ are not unique, they are the most convenient ones for time-frequency estimates.

3. ALMOST DIAGONALIZATION OF PSEUDODIFFERENTIAL OPERATORS

The tools of the previous section have been developed mainly for applications in signal analysis, but in view of the definition of the Weyl transform (1) and of Sjöstrand's class (9), we can taylor these methods to the investigation of pseudo-differential operators. It is now "natural" to study σ^w on time-frequency shifts of a fixed function ("atom") and then study the matrix of σ^w with respect to a Gabor frame. This idea is related to the confinement characterization of $M^{\infty,1}$ [39], but is conceptually much simpler.

3.1. **Almost Diagonalization.** We first establish a simple, but crucial relation between the action of σ^w on time-frequency shifts and the short-time Fourier transform of σ. Recall that

$$j(z_1, z_2) = (z_2, -z_1) \qquad \text{for } z = (z_1, z_2) \in \mathbb{R}^{2d}.$$

Lemma 3.1. *Fix a window* $g \in M_v^1$ *and* $\Phi = W(g, g)$. *Then, for* $\sigma \in M_{v \circ j^{-1}}^{\infty, 1}$,

$$(16) \qquad \left| \langle \sigma^w \pi(z)\varphi, \pi(w)\varphi \rangle \right| = \left| V_\Phi \sigma \Big(\frac{w+z}{2}, j(w-z) \Big) \right| = \left| \mathcal{V}_\Phi \sigma(u, v) \right|,$$

and

$$(17) \qquad |\mathcal{V}_\Phi \sigma(u, v)| = \left| \langle \sigma^w \pi(u - \tfrac{1}{2} j^{-1}(v))g, \pi(u + \tfrac{1}{2} j^{-1}(v))g \rangle \right|$$

for $u, v, w, z \in \mathbb{R}^{2d}$.

Proof. Note that (16) and (17) are well-defined, because the assumption $g \in M_v^1$ implies that $\Phi = W(g, g) \in M_{1 \otimes (v \circ j^{-1})}^1(\mathbb{R}^{2d})$ [11, Prop. 2.5], and so the short-time Fourier transform $V_\Phi \sigma$ makes sense for $\sigma \in M_{v \circ j^{-1}}^{\infty, 1}$.

We use the time-frequency definition of the Weyl transform (1) and the intertwining property Lemma 2.1, then

$$\langle \sigma^w \pi(z)g, \pi(w)g \rangle_{\mathbb{R}^d} = \langle \sigma, W(\pi(w)g, \pi(z)g) \rangle_{\mathbb{R}^{2d}}$$

$$(18) \qquad\qquad\qquad\qquad = \langle \sigma, c M_{j(w-z)} T_{\frac{w+z}{2}} W(g, g) \rangle$$

$$(19) \qquad\qquad\qquad\qquad = \bar{c} \mathcal{V}_{W(g,g)} \sigma \Big(\frac{w+z}{2}, j(w-z) \Big),$$

where c is a phase factor of modulus one.

To obtain (17), we set $u = \frac{w+z}{2}$ and $v = j(w-z)$. Then $w = u + \frac{1}{2} j^{-1}(v)$ and $z = u - \frac{1}{2} j^{-1}(v)$, and reading formula (19) backwards yields (17). ∎

The next result on almost diagonalization is crucial and all properties of the Sjöstrand class will follow easily.

Theorem 3.2 (Almost Diagonalization). *Fix a non-zero* $g \in M_v^1$ *and assume that* $\mathcal{G}(g, \Lambda)$ *is a Gabor frame for* $L^2(\mathbb{R}^d)$. *Then the following properties are equivalent.*
   (i) $\sigma \in M_{v \circ j^{-1}}^{\infty, 1}(\mathbb{R}^{2d})$.
   (ii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function* $H \in L_v^1(\mathbb{R}^{2d})$ *such that*

$$(20) \qquad |\langle \sigma^w \pi(z)g, \pi(w)g \rangle| \leq H(w - z) \quad \forall w, z \in \mathbb{R}^{2d}.$$

   (iii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a sequence* $h \in \ell_v^1(\Lambda)$ *such that*

$$(21) \qquad |\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle| \leq h(\lambda - \mu) \quad \forall \lambda, \mu \in \Lambda.$$

Proof. We first prove the equivalence (i) $\Longleftrightarrow$ (ii) by means of Lemma 3.1.
   (i) $\Longrightarrow$ (ii)   Assume that $\sigma \in M_{v \circ j^{-1}}^{\infty, 1}$ and set

$$H_0(v) = \sup_{u \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(u, v)|$$

By definition of $M^{\infty,1}_{v \circ j^{-1}}$ we have $H_0 \in L^1_{v \circ j^{-1}}(\mathbb{R}^{2d})$, so Lemma 3.1 implies that

$$
\begin{aligned}
|\langle \sigma^w \pi(z)\varphi, \pi(w)\varphi \rangle| &= \left| \mathcal{V}_\Phi \sigma \big(\frac{w+z}{2}, j(w-z) \big) \right| \\
&\leq \sup_{u \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(u, j(w-z))| \\
&= H_0(j(w-z)) \,.
\end{aligned}
\tag{22}
$$

Since $\|H_0 \circ j\|_{L^1_v} = \|H_0\|_{L^1_{v \circ j^{-1}}} < \infty$, we can take $H = H_0 \circ j^{-1} \in L^1_v(\mathbb{R}^{2d})$ as the dominating function in (20).

(ii) $\implies$ (i)   Conversely, assume that $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and that σ^w is almost diagonalized by the time-frequency shifts $\pi(z)$ with dominating function $H \in L^1_v(\mathbb{R}^{2d})$ as in (20). Using the transition formula (9), we find that

$$
\begin{aligned}
|\mathcal{V}_\Phi \sigma(u,v)| &= \left| \left\langle \sigma^w \pi(u - \tfrac{1}{2}j^{-1}(v))g, \pi(u + \tfrac{1}{2}j^{-1}(v))g \right\rangle \right| \\
&\leq H(j^{-1}(v)) \qquad \forall u \in \mathbb{R}^{2d} \,.
\end{aligned}
$$

We conclude that
$$
\int_{\mathbb{R}^{2d}} \sup_{u \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(u,\zeta)| \, v(j^{-1}(\zeta)) \, d\zeta \leq \int_{\mathbb{R}^{2d}} H(v^{-1}(\zeta)) v(j^{-1}(\zeta)) \, d\zeta = \|H\|_{L^1_v} < \infty \,,
\tag{23}
$$

and so $\sigma \in M^{\infty,1}_{v \circ j^{-1}}(\mathbb{R}^{2d})$.

The discrete condition (iii) is similar, but technically more subtle to handle.

(i) $\implies$ (iii)   To show this implication, we use the well-known fact that the short-time Fourier transform of a distribution possesses "nice" local properties, see [11, 21, 44] for various statements and proofs. In particular, if $\sigma \in M^{\infty,1}_{v \circ j^{-1}}$, then $\mathcal{V}_\Phi \sigma \in W(C, \ell^{\infty,1}_{v \circ j^{-1}})(\mathbb{R}^{4d})$ [21, Thm. 12.2.1]. This means the following: let $Q = [-1/2, 1/2]^{2d}$ and define the sequence $a_k, k \in \mathbb{Z}^{2d}$, to be $a_k = \sup_{\zeta \in k+Q} \sup_{z \in \mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(z, \zeta)|$; then

$$
\sum_{k \in \mathbb{Z}^{2d}} a_k \, v(j^{-1}(k)) = \|a\|_{\ell^1_{v \circ j^{-1}}} \leq C \|\sigma\|_{M^{\infty,1}_{v \circ j^{-1}}} < \infty \,.
$$

Using (16) once more, we obtain that

$$
|\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle| = \left| \mathcal{V}_\Phi \sigma \big(\frac{\lambda + \mu}{2}, j(\lambda - \mu) \big) \right| \leq a_k \qquad \text{if } j(\lambda - \mu) \in k + Q \,.
$$

Now set

$$
h(\lambda) = a_k \qquad \text{if } \lambda \in j^{-1}(k + Q) = j^{-1}(k) + Q \,.
$$

Then

$$\sum_{\lambda \in \Lambda} h(\lambda)\, v(\lambda) \;=\; \sum_{k \in \mathbb{Z}^{2d}} \sum_{\lambda \in j^{-1}(k)+Q} a_k\, v(\lambda)$$

(24)
$$\leq \sum_{k \in \mathbb{Z}^{2d}} \sum_{\lambda \in j^{-1}(k)+Q} a_k\, v(j^{-1}(k)) \sup_{u \in Q} v(-u)$$

$$= \; C \max_{k \in \mathbb{Z}^{2d}} \operatorname{card}\{\lambda \in \Lambda : \lambda \in j^{-1}(k)+Q\}|\sum_{k \in \mathbb{Z}^{2d}} a_k\, v(j^{-1}(k))$$

$$\leq \; C'\|\sigma\|_{M_{v \circ j^{-1}}^{\infty,1}}\,.$$

This is (iii) as desired.

(iii) $\implies$ (ii)   To prove this implication, we finally use the hypothesis that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. Since $g \in M_v^1$, the dual window γ is also in M_v^1 by Theorem 2.2. In particular, every time-frequency shift $\pi(u)g$ has the following frame expansion:

(25)
$$\pi(u)g = \sum_{\nu \in \Lambda} \langle \pi(u)g, \pi(\nu)\gamma\rangle \pi(\nu)g\,.$$

If $g, \gamma \in M_v^1$, then by the local properties of short-time Fourier transforms [21, Thm. 12.2.1], we know that $V_\gamma g \in W(C, \ell_v^1)(\mathbb{R}^{2d})$. This means that for every relatively compact set $C \subseteq \mathbb{R}^{2d}$ we have

$$\sum_{\nu \in \Lambda} \sup_{u \in C} |V_\gamma g(\nu + u)| v(\nu) \leq C\|g\|_{M_v^1}$$

In particular, if C is a relatively compact fundamental domain of the lattice Λ and

(26)
$$\alpha(\nu) = \sup_{u \in C} |V_\gamma g(\nu + u)| = \sup_{u \in C} |\langle \pi(-u)g, \pi(\nu)\gamma\rangle|\,,$$

then the sequence α is in $\ell_v^1(\Lambda)$.

Given $z, w \in \mathbb{R}^{2d}$ we can write them uniquely as $w = \lambda + u, z = \mu + u'$ for $\lambda, \mu \in \Lambda$ and $u, u' \in C$. Inserting the expansions (25) and the definition of α in the matrix entries, we find that

$$|\langle \sigma^w \pi(\mu + u')g, \pi(\lambda + u)g\rangle| = |\langle \sigma^w \pi(\mu)\pi(u')g, \pi(\lambda)\pi(u)g\rangle|$$

$$\leq \sum_{\nu, \nu' \in \Lambda} |\langle \sigma^w \pi(\mu + \nu')g, \pi(\lambda + \nu)g\rangle|\, |\langle \pi(u')g, \pi(\nu')\gamma\rangle|\, |\langle \pi(u)g, \pi(\nu)\gamma\rangle|$$

$$\leq \sum_{\nu, \nu' \in \Lambda} h(\lambda + \nu - \mu - \nu')\alpha(\nu')\alpha(\nu)$$

$$= \; (h * \alpha * \check{\alpha})(\lambda - \mu)\,,$$

with $\check{\alpha}(\lambda) = \alpha(-\lambda)$. Since $h \in \ell_v^1$ by hypothesis (iii) and $\alpha \in \ell_v^1$ by construction, we also have $h * \alpha * \check{\alpha} \in \ell_v^1(\Lambda)$.

Now set

$$H(z) = \sum_{\lambda \in \Lambda} (h * \alpha * \check{\alpha})(\lambda)\, \chi_{C-C}(z - \lambda)\,.$$

Then

$$(27) \qquad \|H\|_{L^1_v} \leq \sum_\lambda h * \alpha * \breve\alpha(\lambda)v(\lambda) \; \|\chi_{C-C}\|_{L^1_v} = c\|h * \alpha * \breve\alpha\|_{\ell^1_v} < \infty \,.$$

If $z, w \in \mathbb{R}^{2d}$ with $w = \lambda + u, z = \mu + u'$ for $\lambda, \mu \in \Lambda$ and $u, u' \in C$, then $w - z \in \lambda - \mu + C - C$ and $(h * \alpha * \alpha^*)(\lambda - \mu) \leq H(w - z)$. Combining these observations, we have shown that

$$\left| \langle \sigma^w \pi(z)g, \pi(w)g \rangle \right| \leq (h * \alpha * \breve\alpha)(\lambda - \mu) \leq H(w - z) \,,$$

and this is (ii). ∎

Corollary 3.3. *Under the hypotheses of Theorem 3.2, assume that $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is continuous and satisfies the estimates*

$$\left| \langle T\pi(\mu)g, \pi(\lambda)g \rangle \right| \leq h(\lambda - \mu) \qquad \forall \lambda, \mu \in \Lambda$$

for some $h \in \ell^1_v$. Then $T = \sigma^w$ for some symbol $\sigma \in M^{\infty,1}_{v \circ j^{-1}}$.

Proof. Schwartz's kernel theorem and (1) imply that $T = \sigma^w$ for some distributional symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ (see also [21, Thm. 14.3.5]). Now apply Theorem 3.2. ∎

REMARKS: 1. Motivated by the concept of "confined symbols" [8], Sjöstrand proved that $\sigma \in M^{\infty,1}$ if and only if there exists $h \in \ell^1(\Lambda)$ such that $\|(T_\mu\chi)^w \sigma^w (T_\lambda\chi)^w\|_{L^2 \to L^2} \leq h(\lambda - \mu)$, where $\chi \in \mathcal{S}(\mathbb{R}^{2d})$ satisfies $\sum_{\lambda \in \Lambda} \chi(t - \lambda) = 1$. The equivalence (i) $\iff$ (ii) was also obtained independently by Strohmer [40].

2. Property (ii) says that σ^w preserves the time-frequency localization and that σ^w maps the time-frequency shifts $\pi(z)g$ into functions in M^1_v with a uniform envelope H in the time-frequency plane. This could be rephrased by saying that σ^w maps time-frequency "atoms" into time-frequency "molecules".

3. By property (iii) σ^w is almost diagonalized by the Gabor frame $\mathcal{G}(g, \Lambda)$. It is well-known that certain types of pseudodifferential operators are almost diagonalized with respect to wavelet bases or local Fourier bases [33,35]. What is remarkable in Theorem 3.2 is that the almost diagonalization property actually characterizes a symbol class.

3.2. **Matrix Formulation.** Let us formulate Theorem 3.2 on a more conceptual level. Let $f = \sum_{\mu \in \Lambda} \langle f, \pi(\mu)\gamma \rangle \pi(\mu)g$ be the Gabor expansion of $f \in L^2(\mathbb{R}^d)$, then

$$(28) \qquad C_g(\sigma^w f)(\lambda) = \langle \sigma^w f, \pi(\lambda)g \rangle = \sum_{\mu \in \Lambda} \langle f, \pi(\mu)\gamma \rangle \langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle \,.$$

We therefore define the matrix $M(\sigma)$ associated to the symbol σ with respect to a Gabor frame by the entries

$$(29) \qquad M(\sigma)_{\lambda\mu} = \langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle \,, \qquad \lambda, \mu \in \Lambda \,.$$

With this notation, (28) can be recast as

$$(30) \qquad C_g(\sigma^w f) = M(\sigma)C_\gamma f \,;$$

or as a commutative diagram:

$$
(31) \qquad
\begin{array}{ccc}
L^2(\mathbb{R}^d) & \xrightarrow{\;\sigma^w\;} & L^2(\mathbb{R}^d) \\
\downarrow C_\gamma & & \downarrow C_g \\
\ell^2(\Lambda) & \xrightarrow{\;M(\sigma)\;} & \ell^2(\Lambda)
\end{array}
$$

Lemma 3.4. *If σ^w is bounded on $L^2(\mathbb{R}^d)$, then $M(\sigma)$ is bounded on $\ell^2(\Lambda)$ and maps $\operatorname{ran} C_g$ into $\operatorname{ran} C_g$ with $\ker M(\sigma) \supseteq (\operatorname{ran} C_g)^\perp = \ker C_g^*$.*

Proof. Note that $\operatorname{ran} C_\gamma = \operatorname{ran} C_g$, since $\langle f, \pi(\lambda)\gamma\rangle = \langle f, \pi(\lambda)S^{-1}g\rangle = \langle S^{-1}f, \pi(\lambda)g\rangle$ for all $\lambda \in \Lambda$, or $C_\gamma = C_g S^{-1}$.

Consequently, by the frame property and (31) we have

$$
\|M(\sigma)C_\gamma f\|_2 = \|C_g(\sigma^w f)\|_2 \le C_1\|\sigma^w f\|_2 \le C_2\|f\|_2 \le C_3\|C_g f\|_2 \,,
$$

and so $M(\sigma)$ is bounded from $\operatorname{ran} C_g$ into $\operatorname{ran} C_g$. If $c \in (\operatorname{ran} C_g)^\perp = \ker C_g^*$, then $\sum_{\mu\in\Lambda} c_\mu \pi(\mu)g = 0$, and thus $(M(\sigma)c)(\lambda) = \sum_{\mu\in\Lambda}\langle \sigma^w \pi(\mu)g, \pi(\lambda)g\rangle c_\mu = 0$, i.e., $c \in \ker M(\sigma)$. ∎

Since $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is only a frame, but not a basis, not every matrix A is of the form $M(\sigma)$. It is easy to see that the properties of Lemma 3.4 imply that $A = M(\sigma)$ for some $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.

Next we formalize the properties of the matrices occurring in Theorem 3.2.

Definition 2. We say that a matrix $A = (a_{\lambda\mu})_{\lambda,\mu\in\Lambda}$ belongs to $\mathcal{C}_v = \mathcal{C}_v(\Lambda)$, if there exists a sequence $h \in \ell_v^1(\Lambda)$ such that

$$
(32) \qquad |a_{\lambda\mu}| \le h(\lambda - \mu) \qquad \forall \lambda, \mu \in \Lambda \,.
$$

We endow $\mathcal{C}_v$ with the norm

$$
\begin{aligned}
(33) \qquad \|A\|_{\mathcal{C}_v} &= \inf\{\|h\|_{\ell_v^1} : |a_{\lambda\mu}| \le h(\lambda - \mu), \forall \lambda, \mu \in \Lambda\} \\
&= \sum_{\mu\in\Lambda} \sup_{\lambda\in\Lambda} |a_{\lambda,\lambda-\mu}|\, v(\mu) \,.
\end{aligned}
$$

Since every $A \in \mathcal{C}_v$ is dominated by a convolution operator, the algebra property is evident.

Lemma 3.5. *$\mathcal{C}_v$ is a Banach $*$-algebra.*

REMARK: If $A \in \mathcal{C}_v$, then A is automatically bounded on ℓ_m^p for $1 \le p \le \infty$ and $m \in \mathcal{M}_v$. This follows from the pointwise inequality $|Ac(\lambda)| \le (h * |c|)(\lambda)$ and Young's inequality. If $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then also

$$
(34) \qquad \|Ac\|_{\ell_m^{p,q}} \le \|h * |c|\,\|_{\ell_m^{p,q}} \le \|h\|_{\ell_v^1}\,\|c\|_{\ell_m^{p,q}} \,.
$$

Theorem 3.2 can be recast as follows.

Theorem 3.6. *A symbol σ is in $M_{v\circ j^{-1}}^{\infty,1}$ if and only if $M(\sigma) \in \mathcal{C}_v$ and*

$$
(35) \qquad \|\sigma\|_{M_{v\circ j^{-1}}^{\infty,1}} \asymp \|M(\sigma)\|_{\mathcal{C}_v} \,.
$$

The estimate $\|M(\sigma)\|_{\mathcal{C}_v} \leq C_1 \|\sigma\|_{M^{\infty,1}_{v \circ j^{-1}}}$ is contained in (22), the converse inequality follows by combining (23) and (27).

REMARK: In view of this reformulation it is natural to consider other matrix algebras and study the relation between symbols and the membership of $M(\sigma)$ in a matrix algebra.

4. The Proofs of Sjöstrand's Results

We are now ready to prove Sjöstrand's results in their "natural" context and at the same time we formulate suitable extensions. In place of "hard analysis" we use time-frequency methods, Theorem 3.2, and recent Banach algebra techniques.

Though frames do not enter in the formulation of the results, they are vital in the proofs. To treat all weights in the class $\mathcal{M}_v$, we need to assume as in Theorem 2.2 that the window is chosen from an appropriate space of test functions M_v^1.

4.1. Boundedness.

Theorem 4.1. *If $\sigma \in M^{\infty,1}_{v \circ j^{-1}}$, then σ^w is bounded on $M_m^{p,q}$ for $1 \leq p, q \leq \infty$ and all $m \in \mathcal{M}_v$. The operator norm can be estimated uniformly by*

$$\|\sigma^w\|_{M_m^{p,q} \to M_m^{p,q}} \leq C \|M(\sigma)\|_{\mathcal{C}_v} \asymp \|\sigma\|_{M^{\infty,1}_{v \circ j^{-1}}},$$

with a constant independent of p, q, and m.

Proof. Fix a Gabor frame $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ with window $g \in M^1_{v \circ j^{-1}}$. By Theorem 2.2 also $\gamma \in M^1_{v \circ j^{-1}}$ and the following norms are equivalent on $M_m^{p,q}$: $\|f\|_{M_m^{p,q}} \asymp \|C_g f\|_{\ell_m^{p,q}} \asymp \|C_\gamma f\|_{\ell_m^{p,q}}$ for every $1 \leq p, q \leq \infty$ and $m \in \mathcal{M}_v$.

Now let $f \in M_v^1 \subseteq L^2(\mathbb{R}^d)$ be arbitrary. Applying diagram (31), we estimate the $M_m^{p,q}$-norm of $\sigma^w f$ as follows:

$$\|\sigma^w f\|_{M_m^{p,q}} \leq C_0 \|C_\gamma(\sigma^w f)\|_{\ell_m^{p,q}} = C_0 \|M(\sigma) C_g f\|_{\ell_m^{p,q}}.$$

Since $M(\sigma) \in \mathcal{C}_v$ by Theorem 3.2, $M(\sigma)$ is bounded on $\ell_m^{p,q}$ for $m \in \mathcal{M}_v$ by (34). So we continue the above estimate by

$$\|\sigma^w f\|_{M_m^{p,q}} \leq C_0 \|M(\sigma)\|_{\ell_m^{p,q} \to \ell_m^{p,q}} \|C_g f\|_{\ell_m^{p,q}} \leq C_1 \|M(\sigma)\|_{\mathcal{C}_v} \|f\|_{M_m^{p,q}}.$$

This implies that σ^w is bounded on the closure of M_v^1 in the $M_m^{p,q}$-norm. If $p, q < \infty$, then by density σ^w is bounded on $M_m^{p,q}$. For $p = \infty$ or $q = \infty$, the argument has to be modified as in [5]. ∎

REMARKS: 1. In particular, if $\sigma \in M^{\infty,1}$, then σ^w is bounded on $L^2(\mathbb{R}^d)$ [9,38] and on all $M^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$ [21,23].

2. Theorem 4.1 is a slight improvement over [21, Thm. 14.5.6] where the boundedness on $M_m^{p,q}$ for $m \in \mathcal{M}_v$ required that $\sigma \in M_w^{\infty,1}$ with $w(\zeta) = v(j^{-1}(\zeta)/2)^2 \geq v(j^{-1}(\zeta))$.

Since $S_{0,0}^0 \subseteq M^{\infty,1}$, the Weyl transforms σ^w for $\sigma \in M^{\infty,1}$ cannot be bounded on $L^p(\mathbb{R}^d)$ in general. Using the embeddings $L^p \subseteq M^{p,p'}$ for $1 \leq p \leq 2$ and $L^p \subseteq M^p$ for $2 \leq p \leq \infty$, we obtain an L^p result as follows.

Corollary 4.2. *Assume that $\sigma \in M^{\infty,1}$. If $1 \leq p \leq 2$, then σ^w maps L^p into $M^{p,p'}$, whereas for $2 \leq p \leq \infty$, σ^w maps L^p into M^p.*

4.2. **The Algebra Property.**

Theorem 4.3. *If v is submultiplicative, then $M_v^{\infty,1}$ is a Banach $*$-algebra with respect to the twisted product $\sharp$ and the involution $\sigma \to \bar{\sigma}$.*

Proof. It is convenient to use a tight Gabor frame $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ with $\gamma = g \in M_v^1$ as in (14). By using (31) twice, we obtain that

$$
\begin{aligned}
M(\sigma \sharp \tau)C_g f &= C_g((\sigma \sharp \tau)^w f) = C_g(\sigma^w \tau^w f) \\
&= M(\sigma)\big(C_g(\tau^w f)\big) = M(\sigma)M(\tau)C_g f .
\end{aligned}
$$

Therefore the operators $M(\sigma \sharp \tau)$ and $M(\sigma)M(\tau)$ coincide on $\operatorname{ran} C_g$. Since $M(\sigma)|_{(\operatorname{ran} C_g)^\perp} = 0$ for all $\sigma \in M^{\infty,1}$ by Lemma 3.4, we obtain that

$$
(36) \qquad\qquad M(\sigma \sharp \tau) = M(\sigma)M(\tau)
$$

as an identity of matrices (on ℓ^2).

Now, if $\sigma, \tau \in M_v^{\infty,1}$, then $M(\sigma), M(\tau) \in \mathcal{C}_{v \circ j}$ by Theorem 3.6. By the algebra property of $\mathcal{C}_{v \circ j}$ we have $M(\sigma)M(\tau) \in \mathcal{C}_{v \circ j}$, and once again by Theorem 3.6 we have $M(\sigma \sharp \tau) \in \mathcal{C}_{v \circ j}$ with the norm estimate

$$
\|\sigma \sharp \tau\|_{M_v^{\infty,1}} \leq C_0 \|M(\sigma \sharp \tau)\|_{\mathcal{C}_{v \circ j}} \leq C_0 \|M(\sigma)\|_{\mathcal{C}_{v \circ j}} \|M(\tau)\|_{\mathcal{C}_{v \circ j}} \leq C_1 \|\sigma\|_{M_v^{\infty,1}} \|\tau\|_{M_v^{\infty,1}} .
$$

$\blacksquare$

Compare [22, 38, 39, 42] for other proofs.

4.3. **Wiener Property of Sjöstrand's Class.** For the Wiener property we start with two results about the Banach algebra $\mathcal{C}_v$.

Theorem 4.4. *Assume that v is a submultiplicative weight satisfying the GRS-condition*

$$
(37) \qquad\qquad \lim_{n \to \infty} v(nz)^{1/n} = 1 \quad \forall z \in \mathbb{R}^{2d} .
$$

If $A \in \mathcal{C}_v$ and A is invertible on $\ell^2(\mathbb{Z}^d)$, then $A^{-1} \in \mathcal{C}_v$. As a consequence

$$
(38) \qquad\qquad \operatorname{Sp}_{\mathcal{B}(\ell^2)}(A) = \operatorname{Sp}_{\mathcal{C}_v}(A)
$$

for all $A \in \mathcal{C}_v$, where $\operatorname{Sp}_{\mathcal{A}}(A)$ denotes the spectrum of A in the algebra $\mathcal{A}$.

Originally, this important result was proved by Baskakov [1, 2] in several papers, and by Sjöstrand [39] for the unweighted case $v \equiv 1$.

Recall that an operator $A : \ell^2 \to \ell^2$ is pseudo-invertible, if there exists a closed subspace $\mathcal{R} \subseteq \ell^2$, such that A is invertible on $\mathcal{R}$ and $\ker A = \mathcal{R}^\perp$. The unique operator $A^\dagger$ that satisfies $A^\dagger A h = A A^\dagger h = h$ for $h \in \mathcal{R}$ and $\ker A^\dagger = \mathcal{R}^\perp$ is called the (Moore-Penrose) pseudo-inverse of A. The following lemma is borrowed from [19].

Lemma 4.5 (Pseudoinverses). *If $A \in \mathcal{C}_v$ has a (Moore-Penrose) pseudoinverse $A^\dagger$, then $A^\dagger \in \mathcal{C}_v$.*

Proof. By means of the Riesz functional calculus [36] the pseudoinverse can be written as

$$A^\dagger = \frac{1}{2\pi i} \int_C \frac{1}{z} (z\mathrm{I} - A)^{-1} \, dz \, ,$$

where C is a suitable path surrounding $\mathrm{Sp}_{\mathcal{B}(\ell^2)}(A) \setminus \{0\}$. By (38) this formula make sense in $\mathcal{C}_v$, and consequently $A^\dagger \in \mathcal{C}_v$. ∎

Theorem 4.6. *Assume that v satisfies the GRS-condition $\lim_{n\to\infty} v(nx)^{1/n} = 1, \forall x \in \mathbb{R}^{2d}$. If $\sigma \in M_v^{\infty,1}(\mathbb{R}^{2d})$ and σ^w is invertible on $L^2(\mathbb{R}^d)$, then $(\sigma^w)^{-1} = \tau^w$ for some $\tau \in M_v^{\infty,1}$.*

Proof. Again, we use a tight Gabor frame $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ with $g = \gamma \in M_v^1$ as in (14) for the analysis of the Weyl transform.

Let $\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ be the unique distribution such that $\tau^w = (\sigma^w)^{-1}$. Then the matrix $M(\tau)$ is bounded on ℓ^2 and maps $\mathrm{ran}\, C_g$ into $\mathrm{ran}\, C_g$ with $\ker M(\tau) \subseteq (\mathrm{ran}\, C_g)^\perp$ (by Lemma 3.4).

We show that $M(\tau)$ is the pseudo-inverse of $M(\sigma)$. Let $c = C_g f \in \mathrm{ran}\, C_g$, then

$$M(\tau)M(\sigma)C_g f = M(\tau)C_g(\sigma^w f) = C_g(\tau^w \sigma^w f) = C_g f \, ,$$

and likewise $M(\sigma)M(\tau) = \mathrm{I}_{\mathrm{ran}\, C_g}$. Since $\ker M(\sigma), \ker M(\tau) \subseteq (\mathrm{ran}\, C_g)^\perp$, we conclude that $M(\tau) = M(\sigma)^\dagger$.

By Theorem 3.2 the hypothesis $\sigma \in M_v^{\infty,1}$ implies that $M(\sigma)$ belongs to the matrix algebra $\mathcal{C}_{v\circ j}$. Consequently by Lemma 4.5, we also have $M(\tau) = M(\sigma)^\dagger \in \mathcal{C}_{v\circ j}$. Using Theorem 3.2 again, we conclude that $\tau \in M_v^{\infty,1}$. This finishes the proof of the Wiener property. ∎

It can be shown that Theorem 4.6 is false, when v does not satisfy (37). Thus the GRS-condition is sharp.

Corollary 4.7 (Spectral Invariance on Modulation Spaces). *If $\sigma \in M_{v\circ j^{-1}}^{\infty,1}$ and σ^w is invertible on $L^2(\mathbb{R}^d)$, then σ^w is invertible simulaneously on all modulation spaces $M_m^{p,q}(\mathbb{R}^d)$, where $1 \leq p, q \leq \infty$ and $m \in \mathcal{M}_v$.*

Proof. By Theorem 4.6 $(\sigma^w)^{-1} = \tau^w$ for some $\tau \in M_{v\circ j^{-1}}^{\infty,1}$ and then by Theorem 4.1 τ^w is bounded on $M_m^{p,q}$ for the range of p, q and m specified. Since $\sigma^w \tau^w = \tau^w \sigma^w = \mathrm{I}$ on M_v^1, this factorization extends by density to all of $M_m^{p,q}$. Thus $\tau^w = (\sigma^w)^{-1}$ on $M_m^{p,q}$. ∎

REMARKS: 1. It is known that $M^{\infty,1}$ is invariant under convolution with "chirps" $e^{it\cdot Ct}$ for any symmetric real-valued $d \times d$-matrix C [21, 38]. As a consequence, the properties of the symbol class $M^{\infty,1}$ carry over to other calculi of pseudodifferential operators, in particular to the Kohn–Nirenberg correspondence [38, 42].

2. Translation and modulation operators can be defined on arbitrary locally compact abelian groups (LCA groups), and consequently, modulation spaces and

the Kohn–Nirenberg correspondence are well-defined on LCA groups in place of $\mathbb{R}^d$. Therefore Sjöstrand's results should hold in the general context of LCA groups, but it is clear that the methods of classical analysis can no longer be applied, whereas it is plausible that time-frequency methods can be generalized. For instance, it is not hard to verify that the matrix algebra $\mathcal{C}_v$ for $v \equiv 1$ coincides with $M^{\infty,1}(\mathbb{Z}^d \times \mathbb{T}^d)$. Thus Theorem 4.4 says that the Wiener property holds for the modulation space $\mathcal{C} = M^{\infty,1}(\mathbb{Z}^d \times \mathbb{T}^d)$. Therefore we conjecture that Theorem 4.6 holds for $M^{\infty,1}(\mathcal{G} \times \widehat{\mathcal{G}})$ for an arbitrary LCA group $\mathcal{G}$, and will pursue this question elsewhere.

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